Math 251
Exam 2
Spring 2016

Name Key

Show All Work

1) If $\mathbf{r}(t) = < 4t, \sin 3t, \cos 3t >$, find

a) $\mathbf{T}(t)$

$\vec{r}'(t) = \langle 4, 3\cos 3t, -3\sin 3t \rangle$

$|\vec{r}'(t)| = \sqrt{16 + 9\cos^2 3t + 9\sin^2 3t} = \sqrt{16+9} = \sqrt{25} = 5$

$\boxed{\vec{T}(t) = \langle \frac{4}{5}, \frac{3}{5}\cos 3t, -\frac{3}{5}\sin 3t \rangle}$

b) N(t)

$\vec{T}'(t) = \langle 0, -\frac{9}{5}\sin 3t, -\frac{9}{5}\cos 3t \rangle$

$|\vec{T}'(t)| = \sqrt{0 + \frac{81}{25}\sin^2 3t + \frac{81}{25}\cos^2 3t} = \sqrt{\frac{81}{25}} = \frac{9}{5}$

$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle 0, -\sin 3t, -\cos 3t \rangle}$

c) the curvature κ

$\boxed{\kappa = \frac{|T'|}{|r'|} = \frac{9/5}{5} = \frac{9}{25}}$

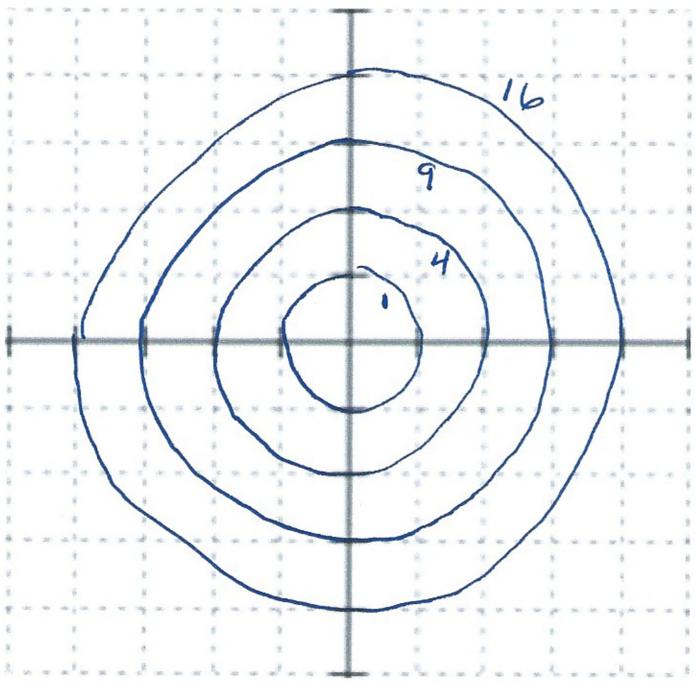
d) The length of the curve when $0 \leq t \leq 3$

$\int_0^3 |\vec{r}'(t)|\, dt = \int_0^3 5\, dt = 5t \Big|_0^3 = 15$

2) If $\mathbf{r}(t)$ is the position function of a particle what formula's would you use to compute the tangential and normal components of acceleration.

$a_T = \dfrac{r' \cdot r''}{|r'|}$ $\qquad$ $a_N = \dfrac{|r' \times r''|}{|r'|}$

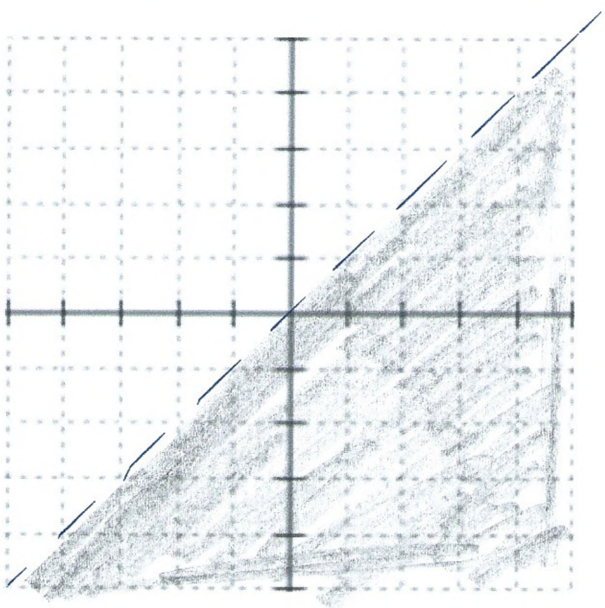
3) Draw a contour map of $f(x,y) = x^2 + y^2$ LABEL IT!



$K = x^2 + y^2$

circles of radius $\sqrt{K}$

3) Find and sketch the domain of f(x, y) = $f(x,y) = \frac{1}{\sqrt{x-y}}$



$x - y > 0$

$x > y$

$\underline{y < x}$

5) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

on the x-axis

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

on the y-axis

$$\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

limit D.N.E.

There is not one number that could be $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

6) Find all first and second partial derivatives of $f(x,y) = 3x^2y^2 - 2x + \frac{x}{y}$

$$f_x = 6xy^2 - 2 + \frac{1}{y} \qquad f_y = 6x^2y - \frac{x}{y^2}$$

$$f_{xx} = 6y^2 \qquad f_{yy} = 6x^2 + \frac{2x}{y^3}$$

$$f_{xy} = 12xy - \frac{1}{y^2} \qquad f_{yx} = 12xy - \frac{1}{y^2}$$

7) If $e^x + y = \tan(xy)$, find dy/dx without using implicit differentiation

$$e^x + y - \tan(xy) = 0$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{e^x - y\sec^2(xy)}{1 - x\sec^2(xy)} = \frac{y\sec^2(xy) - e^x}{1 - x\sec^2(xy)}$$

8) a) Find the equation of the tangent plane to the surface $z = (4 - x^2 + y^2)^{3/2}$ at the point (1, 1, 8).

$\frac{dz}{dx} = \frac{3}{2}(4-x^2+y^2)^{1/2} \cdot (-2x)$ $\qquad$ $\left.\frac{dz}{dx}\right|_{(1,1)} = \frac{3}{2} \cdot 2 \cdot -2 = -6$

$\frac{dz}{dy} = \frac{3}{2}(4-x^2+y^2)^{1/2} \cdot 2y$ $\qquad$ $\left.\frac{dz}{dy}\right|_{(1,1)} = \frac{3}{2} \cdot 2 \cdot 2 = 6$

$z - z_0 = f_x(x-x_0) + f_y(y-y_0)$

$z - 8 = -6(x-1) + 6(y-1)$ ← OK ANSWERS etc.

$z - 8 = -6x + 6 + 6y - 6$

$z = -6x + 6y + 8$

b) Find the linearization of the above function at the above point.

$z = -6x + 6y + 8$

9) Let $f(x, y) = \sin(2x + 6y)$ and let P = (-3, 1)

a) Find the gradient of f at P.

$\nabla f = \langle 2\cos(2x+6y), 6\cos(2x+6y) \rangle$

$\left.\nabla f\right|_{(-3,1)} = \langle 2\cos 0, 6\cos 0 \rangle = \langle 2, 6 \rangle$

b) Find the maximum rate of change of f at the given point and the direction in which it occurs.

max rate of change $= |\nabla f| = \sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$

it occurs in the direction $\langle 2, 6 \rangle$

c) Find the directional derivative of f(x, y) in the direction of v = <3, 4> at P.

$\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_u f = \nabla f \cdot u$

$= \langle 2, 6 \rangle \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= \frac{6}{5} + \frac{24}{5} = \frac{30}{5} = 6$

10) Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^3 + y^2 - 3x + 4$$

$\frac{df}{dx} = 3x^2 - 3$

$\frac{df}{dy} = 2y$

$\frac{df}{dx} = 0$ if $3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$\frac{df}{dy} = 0$ if $y = 0$

critical points $(1, 0)$ and $(-1, 0)$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 6x \cdot 2 - (0)^2 = 12x$

$D(1,0) = 12$ and $f_{xx} > 0$ concave up

there is a min at $(1,0)$

min is $f(1,0) = 1 + 0 - 3 + 4 = 2$

$D(-1,0) = -12$ there is a saddle point at $(-1,0)$

saddle point is $(-1, 0, 6)$

since $f(-1,0) = -1 + 0 + 3 + 4 = 6$